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MDE/M-23 **4078**

REAL ANALYSIS

Paper–II : MM–402

Time Allowed : 3 Hours] [Maximum Marks : 80

Note : Attempt **five** questions in all, selecting at least **one** question from each Unit.

UNIT–I

1. (a) If $f \in \mathbb{R}$ on [a, b] and if there is a differentiable function F on [a, b] such that $F^1 = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$ 8

(b) Suppose $f \in \mathbb{R}(\alpha)$ on [a, b], $m \le f \le M$, ϕ is continuous on [m, M], and $h(x) = \phi(f(x))$ on [a, b]. Then $h \in \mathbb{R}(\alpha)$ on [a, b]. 8

2. (a) Define norm of a function $F \in \zeta(x)$. Prove that $\zeta(x)$ is complete metric space. 8

(b) For n = 1, 2, 3, ..., x real, put

$$F_n(x) = \frac{x}{1+nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$F^1(x) = \lim_{n\to\infty} F_n^1(x)$$ 8

3. (a) Suppose the series

$$\sum_{n=0}^{\infty} C_n x^n$$

converges for $|x| < R$, and define

$$F(x) = \sum_{n=0}^{\infty} C_n x^n \quad (|x| < R).$$

Then $\sum_{n=0}^{\infty} C_n x^n$ converges uniformly on $[-R+\in, R-\in]$, no matter which $\in > 0$ is chosen. The function F is continuous and differentiable in (–R, R), and $F^1(x) = \sum_{n=1}^{\infty} nC_n x^{n-1} \quad (|x| < R)$. 8

(b) (i) Show that $\int_0^4 x\,d([x]-x) = \frac{3}{2}$. Where [x] is the greatest integer not exceeding x.

(ii) $\int_{\pi}^{2\pi} \sin x \, d(\cos x) = \frac{-\pi}{2}$. 8

UNIT–II

4. (a) Suppose F maps an open set $E \subset R^n$ into R^m. Then $F \in \ell^1(E)$ if and only if the partial derivatives D_jF_i exist and are continuous on E for $1 \le i \le m$, $1 \le j \le n$. 8

(b) If $F \in \ell(R^n)$ and the support of F lies in K, then $F = \sum_{i=1}^{s} \psi_i F$. Each $\psi_i F$ has its support in some V_α. 8

5. State and prove the Inverse function theorem. 16

UNIT–III

6. (a) Let $\{E_i\}$ be on infinite decreasing sequence of measurable sets; that is, a sequence with $E_{i+1} \subset E_i$ for each $i \in \mathbb{N}$. Let $m(E_i) < \infty$ for at least one $i \in \mathbb{N}$. Then

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m(E_n).$$ 8

(b) Let g be an integrable function on E and let $\{F_n\}$ be a sequence of measurable functions such that $|F_n| \le g$ on E and $\lim_{n \to \infty} F_n = f$ a.e. on E. Then $\int_E F = \lim_{n \to \infty} \int_E F_n$. 8

7. (a) Evaluate the integral (Lebesgue) for the function $F : [0, \infty] \to \mathbb{R}$ given by

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \ne 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$ 8

(b) Let F be a bounded function defined on [a, b]. If F is Riemann integrable over [a, b], then it is Lebesgue integrable and $R\int_a^b f(x)\,dx = L\int_a^b f(x)\,dx$. 8

UNIT–IV

8. (a) Let F be a function of bounded variation on [a, b]. Then F is continuous at a point in [a, b] if and only if its variation function V_f is so. 8

(b) Let F be a bounded and measurable function defined on [a, b]. If $F(x) = \int_a^x f(t)\,dt + F(a)$, then $F^1(x) = f(x)$ a.e. in [a, b]. 8

9. (a) If F is an absolutely continuous function on [a, b], then F is an indefinite integral of its derivative; more precisely:

$$F(x) = \int_a^x f(t)\,dt + C,$$

where $f = F^1$ and C is a constant. 8

(b) For p, $1 \le p < \infty$, prove that :

(i) if $g \in L^p$ and $|f| < |g|$, then

$f \in L^p$

(ii) if $F, g \in L^p$, then

$fg \in L^{\frac{p}{2}}$. 8

10. Show that the normed spaces $L^p, 1 \le p \le \infty$, are complete. 16